Journal of Applied Mathematics and Mechanics 73 (2009) 156–161

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The dynamics of a Chaplygin sleigh☆

A.V. Borisov, I.S. Mamayev

Izhevsk, Russia

ARTICLE INFO

Article history:
Received 22 September 2008

ABSTRACT

The problem of the motion of a Chaplygin sleigh on horizontal and inclined surfaces is considered. The possibility of representing the equations of motion in Hamiltonian form and of integration using Liouville's theorem (with a redundant algebra of integrals) is investigated. The asymptotics for the rectilinear uniformly accelerated sliding of a sleigh along the line of steepest descent are determined in the case of an inclined plane. The zones in the plane of the initial conditions, corresponding to a different behaviour of the sleigh, are constructed using numerical calculations. The boundaries of these domains are of a complex fractal nature, which enables a conclusion to be drawn concerning the probable character from of the dynamic behaviour.

1. The equations of motion

Chaplygin[1] considered the motion of a rigid body supported on a plane by two posts with absolutely smooth ends and by a sharp small wheel (a disc or a blade) such that the body cannot move perpendicular to the plane of the small wheel (a Chaplygin sleigh). We select two systems of coordinates, a fixed Oxy system and a system $O'\xi\eta$ which is rigidly connected with the body and has its origin O' located at the intersection of the straight line passing through the contact point of the small wheel Q perpendicularly to its plane and the straight line passing through the centre of mass C parallel to the plane of the wheel (Fig. 1); a is the distance from the point O' to the centre of mass and b is the distance from the point O' to the contact point of the wheel. Suppose ω is the angular velocity of the body and v_1 and v_2 are the projections of the velocity of the point Q' onto the moving axes. The equation of the constraint (which expresses the condition that the projections of the velocity of the point Q onto the $O'\eta$ axis are equal to zero) then has the form

$$v_2 = 0 \tag{1.1}$$

The equations of motion of a Chaplygin sleigh in a potential field with potential $U(x,y,\varphi)$can be obtained by Hamel,s method

$$\begin{gathered} m\dot{v}_1 = ma\omega^2 - \frac{\partial U}{\partial x}\cos\varphi - \frac{\partial U}{\partial y}\sin\varphi, \quad (I + ma^2)\dot{\omega} = -ma\omega v_1 - \frac{\partial U}{\partial \varphi} \\ \dot{\varphi} = \omega, \quad \dot{x} = v_1\cos\varphi, \quad \dot{y} = v_1\sin\varphi \end{gathered} \tag{1.2}$$

where m is the mass of the sleigh, I is its moment of inertia about the centre of mass, x and y are the coordinates of the point O' in the fixed system of coordinates and ν is the angle of rotation of the moving axes (Fig. 1)

Equations (1.2) have the energy integral

$$E = (mv_1^2 + (I + ma^2)\omega^2)/2 + U(x, y, \varphi) \tag{1.3}$$

Note that the parameter b does not appear in the equations of motion with this special choice of the moving axes.

We will consider special cases of system (1.2) for which additional symmetries emerge.

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 2, pp. 219–225, 2009.

E-mail addresses: borisov@ics.rg.ru (A.V. Borisov), mamaev@ics.org.ru (I.S. Mamayev).

0021-8928/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2009.04.005

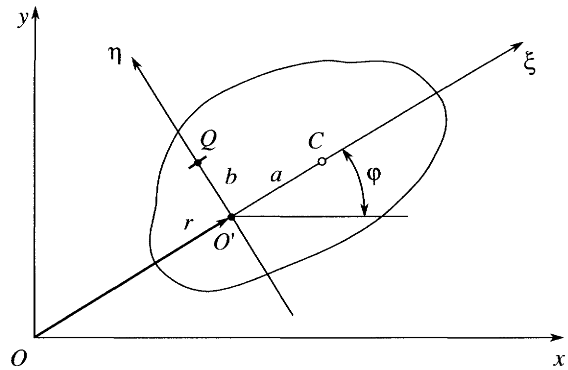

Fig. 1.

2. Motion on a horizontal plane

Hamiltonian form and integrability. When there is no external field, system (1.2) possesses an invariant measure $\rho d\upsilon_1 \wedge d\omega \wedge d\varphi \wedge dx \wedge dy$ with a singular density $\rho = \omega^{-1}$. At the same time, the equations describing the evolution of the projections of the velocity υ_1 and the angular velocity ω separate, and they can be represented in Hamiltonian form in a solvable two-dimensional Lie algebra

$$\dot{\omega} = \{\omega, H\}, \quad \dot{\upsilon}_1 = \{\upsilon_1, H\}$$

$$H = \frac{m}{2}(\upsilon_1^2 + a^2A^2\omega^2), \quad A^2 = 1 + \frac{I}{ma^2}, \quad \{\omega, \upsilon_1\} = -\frac{\omega}{aA^2}$$

This was noted earlier[2] and the analogy between the motion of a Chaplygin sleigh and the dynamics of a two-dimensional open Toda chain has been pointed out.

It is found that system (1.2) can also be represented in Hamiltonian form for a wide choice of the variables υ_1, ω, x, y, φ such that the following theorem, which is proved by direct verification of the equations of motion and the Jacobi identity, holds.

Theorem. If $U = 0$, system (1.2) can be represented in the Hamiltonian form

$$\dot{\xi}_i = \{\xi_i, H\}, \quad \boldsymbol{\xi} = (\upsilon_1, \omega, x, y, \varphi)$$

with Hamiltonian

$$H = \frac{m}{2}(\upsilon_1^2 + a^2A^2\omega^2), \quad A^2 = 1 + \frac{I}{ma^2},$$

and the non-linear Poisson bracket (containing the arbitrary constant Λ)

$$\{\omega, \upsilon_1\} = -\frac{\omega}{aA^2}, \quad \{\omega, \varphi\} = -\frac{\omega^2}{2H}, \quad \{\omega, x\} = -\frac{\omega\cos\varphi}{2H}, \quad \{\omega, y\} = -\frac{\upsilon_1\omega\sin\varphi}{2H}$$

$$\{\upsilon_1, \varphi\} = -\frac{\upsilon_1\omega}{2H}, \quad \{\upsilon_2, x\} = -\frac{\upsilon_1^2\cos\varphi}{2H}, \quad \{\upsilon_1, y\} = -\frac{\upsilon_1^2\sin\varphi}{2H}; \quad \{x, y\} = \Lambda = \text{const} \tag{2.1}$$

Remark 1. Although it was shown by Liouville that a system can be represented in Hamiltonian form when the number of variables is doubled, such a representation is seldomly used in practice. This is due to the fact that the resulting Hamiltonian is degenerate with respect to the momenta. A problem on the possibility of writing the equations of non-holonomic systems in Hamiltonian form without doubling the number of variables while preserving the non-degeneracy of the Hamiltonian has been formulated by V. S. Novoselov, V. V. Kozlov and J. Duistermaat. It is precisely this formulation of the problem which enables us to use the well developed technique of the analysis of Hamiltonian systems for investigating non-holonomic systems.

When $\Lambda \neq 0$, the rank of the Poisson structure is equal to four and the unique central function has the form

$$F = \frac{\varphi}{A} - \operatorname{arctg}\frac{\upsilon_1}{aA\omega} \tag{2.2}$$

Moreover, there are two (non-involute) integrals which determine the trajectory of the body on the plane:

$$\Phi_x = x - aA\int^{\varphi}\cos\zeta\operatorname{ctg}\xi(\zeta)d\zeta, \quad \Phi_y = y - aA\int^{\varphi}\sin\zeta\operatorname{ctg}\xi(\zeta)d\zeta; \quad \{\Phi_x, \Phi_y\} = \Lambda \tag{2.3}$$

where

$$\xi(\zeta) = \frac{\varphi - \zeta}{A} + \operatorname{arctg}\frac{aA\omega}{\upsilon_1}$$

Hence, when $U=0$, system (1.2) possesses a redundant set of non-commutative integrals. When $\Lambda=0$, the rank of the Poisson bracket (2.1) is equal to two, and Φ_x and Φ_y become Casimir functions.

Relations (2.2) and (2.3) define an integrable mapping of scattering, in particular:

$$\Delta\varphi|_{-\infty}^{+\infty} = \pi A$$

In the second special case, when $a=0$, there is a standard invariant measure $dv_1 \wedge d\omega \wedge d\varphi \wedge dx \wedge dy$.

Suppose $a=0$ and $\partial U/\partial y = 0$. Then, the system which describes the evolution of the variables v_1, ω, x, φ is reparable. On changing the time and spatial variables in it

$$\cos\varphi dt = d\tau, \quad x_1 = x, \quad x_2 = \sin\varphi$$

we obtain the equations

$$m\frac{d\upsilon_1}{d\tau} = -\frac{\partial U}{\partial x_1}, \quad J\frac{d\omega}{d\tau} = -\frac{\partial U}{\partial x_2}, \quad J = I + ma_1^2, \quad \frac{dx_1}{d\tau} = \upsilon_1, \quad \frac{dx_2}{d\tau} = \omega \tag{2.4}$$

which, obviously, after a Legendre transformation, are written in Hamiltonian form.

Note that the changed time variable has singularities at the points

$$\cos\varphi = 0 \tag{2.5}$$

Hence, representation (2.4) only holds in an open domain of the phase space from which points satisfying Eq. (2.5) have been excluded.

Remark 2. Chaplygin represented Eqs (2.1) in canonical Hamiltonian form using quasi-coordinates. They are of the fourth order and are convenient for using perturbation theory.[3] Chaplygin used this system to illustrate the reducing factor method which he also applied to more complex problems.[4]

A Chaplygin sleigh with a torque. Another case, which leads to system (1.2), is obtained if the body is balanced and moves on a horizontal plane but has a torque ($\partial U/\partial x = \partial U/\partial y = 0, \partial U/\partial\varphi \neq 0$). In this case, the system of three equations

$$\dot{\upsilon}_1 = a\omega^2, \quad J\dot{\omega} = -ma\omega\upsilon_1 - \partial U/\partial\varphi, \quad \dot{\varphi} = \omega$$

which has an energy integral

$$E = (m\upsilon_1^2 + J\omega^2)/2 + U(\varphi)$$

is separable.

This system can be reduced to just one second-order equation for the function $v_1(\varphi)$

$$Jd^2\upsilon_1/d\varphi^2 = -ma^2\upsilon_1 - a^2(\partial U/\partial\varphi)(d\upsilon_1/d\varphi)^{-1}$$

3. Motion on an inclined plane

We direct the *Ox* axis along the line of steepest descent, and, then,

$$U = m\mu(x + a\cos\varphi), \quad \mu = g\sin\chi \tag{3.1}$$

where χ is the angle of inclination of the plane to the horizontal.

If $a=0$, the equation is trivially integrable and there are four integrals which determine the law of motion and the trajectory:

$$\omega = \text{const}, \quad \upsilon_1 + \frac{\mu}{\omega}\sin\varphi = \text{const}$$

$$x - \frac{1}{\omega^2}\left(\upsilon_1\omega\sin\varphi + \frac{\mu}{2}\sin^2\varphi\right) = \text{const}, \quad y + \frac{1}{\omega^2}\left(\upsilon_1\omega\cos\varphi + \frac{\mu}{2}\cos\varphi + \frac{\mu}{2}\cos\varphi\sin\varphi\right) = \text{const}$$

The body does not slide down but drifts in a horizontal direction while, if the body is released without an initial impetus $v_1(0)=0$, it moves along a cycloid.[5]

The qualitative characteristics of the motion of a Chaplygin sleigh on an inclined plane have been studied[3] by the method of averaging using a canonical Hamiltonian form of the equations proposed by Chaplygin[1] which contains a quasi-coordinate.

System (1.2) with the potential (3.1) has the obvious particular solutions of the form

$$
\begin{aligned}
&1)\ \omega = 0, \quad \upsilon_1 = u_0 - \mu t, \quad \varphi = 0, \quad y = \text{const}, \quad x = u_0 t - \mu t^2/2 \\
&2)\ \omega = 0, \quad \upsilon_1 = u_0 + \mu t, \quad \varphi = \pi, \quad y = \text{const}, \quad x = -u_0 t - \mu t^2/2 \\
&u_0 = \upsilon_1\big|_{t=0}
\end{aligned}
\tag{3.2}
$$

which correspond to the uniformly accelerated sliding of the sleigh along the straight line of steepest descent such that, at the same time, the blade remains parallel to this line.

The equations of motion do not change under the substitution

$$
a \to -a, \quad \varphi \to \pi + \varphi, \quad \upsilon_1 \to -\upsilon_1
$$

We therefore, without loss of generality, put $a > 0$ and investigate the stability of solutions (3.2) (with respect to past of the variables) in greater detail. We carry out a change in the time and a corresponding change in the velocities using the formulae

$$
tdt = d\tau, \quad \upsilon_1 = t\bar{u}, \quad \omega = t\bar{\omega}
\tag{3.3}
$$

In the new time, we obtain the non-autonomous system

$$
\begin{aligned}
&\frac{d\bar{u}}{d\tau} = a\bar{\omega}^2 - \frac{1}{2\tau}(\bar{u} + \mu\cos\varphi), \quad \frac{d\bar{\omega}}{d\tau} = -\frac{\bar{u}\bar{\omega}}{aA^2} - \frac{1}{2\tau}\left(\bar{\omega} - \frac{\mu}{aA^2}\sin\varphi\right), \quad \frac{d\varphi}{d\tau} = \bar{\omega} \\
&\frac{dx}{d\tau} = \bar{u}\cos\varphi, \quad \frac{d\bar{y}}{d\tau} = \bar{u}\sin\varphi
\end{aligned}
\tag{3.4}
$$

The first three equations form a closed system and the fixed points of this system $\bar{u} = -\mu$, $\bar{\omega} = 0$, $\varphi = 0$ and $\bar{u} = \mu$, $\bar{\omega} = 0$, $\varphi = \pi$ correspond to the solutions (3.2) respectively. Linearizing system (3.4) close to them, we obtain

$$
\frac{d\Delta\bar{u}}{d\tau} = -\frac{1}{2\tau}\Delta\bar{u}, \quad \frac{d\Delta\bar{\omega}}{d\tau} = \left(\pm\nu^2 - \frac{1}{2\tau}\right)\Delta\bar{\omega} \pm \frac{\nu^2}{2\tau}\Delta\varphi, \quad \frac{d\Delta\varphi}{d\tau} = \Delta\bar{\omega}; \quad \nu^2 = \frac{\mu}{aA^2}
\tag{3.5}
$$

where the upper sign corresponds to the first solution and the lower sign to the second solution. The solutions of this system have the form

$$
\begin{aligned}
&1)\ \Delta\bar{u} = \frac{C}{\sqrt{\tau}}, \quad \Delta\varphi = \exp(-\nu^2\tau)\left(A + B\frac{\sqrt{\pi}}{\nu}\operatorname{erfi}(\nu\sqrt{\tau})\right) \\
&\Delta\bar{\omega} = \frac{B}{\sqrt{\tau}} - \nu^2\exp(-\nu^2\tau)\left(A + B\frac{\sqrt{\pi}}{\nu}\operatorname{erfi}(\nu\sqrt{\tau})\right) \\
&2)\ \Delta\bar{u} = \frac{C}{\sqrt{\tau}}, \quad \Delta\varphi = \exp(\nu^2\tau)\left(A + B\frac{\sqrt{\pi}}{\nu}\operatorname{erf}(\nu\sqrt{\tau})\right) \\
&\Delta\bar{\omega} = \frac{B}{\sqrt{\tau}} + \nu^2\exp(-\nu^2\tau)\left(A + B\frac{\sqrt{\pi}}{\nu}\operatorname{erf}(\nu\sqrt{\tau})\right)
\end{aligned}
$$

where A, B and C are constants of integration and erf and erfi are the real and imaginary error functions

$$
\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt, \quad \operatorname{erfi}(z) = -i\operatorname{erf}(iz) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(t^2)dt
$$

In the linear approximation, we therefore obtain that the fixed point $\bar{u} = -\mu$, $\bar{\omega} = 0$, $\varphi = 0$ of system (3.4) is unstable and the fixed point $\bar{u} = -\mu$, $\bar{\omega} = 0$, $\varphi = \pi$ is asymptotically stable.

We will show that, in this case, Lyapunov asymptotic stability also holds. After the change of variables

$$
\bar{u} = \rho\cos\psi, \quad aA\bar{\omega} = \rho\sin\psi
$$

the equations of motion (3.4) can be represented in the form

$$\frac{d\rho}{d\tau} = -\frac{\rho}{2\tau} - \frac{\mu(A\cos\varphi\cos\psi - \sin\varphi\sin\psi)}{2A\tau}$$

$$\frac{d\psi}{d\tau} = -\rho\frac{\sin\psi}{aA^2} + \frac{\mu(A\cos\varphi\sin\psi + \sin\varphi\cos\psi)}{2A\tau\rho}$$

$$\frac{d\varphi}{d\tau} = \rho\frac{\sin\psi}{aA}$$

The fixed point investigated has the coordinates

$$\rho = \mu, \quad \psi = 0, \quad \varphi = \pi$$

Suppose

$$X = \rho - \mu, \quad Y = \psi, \quad Z = \varphi - \pi$$

We now select the Lyapunov function in the neighbourhood of the fixed point in the form of the positive-definite quadratic form

$$F = \frac{1}{2}\left(X^2 + Y^2 + \left(Y + \frac{Z}{A}\right)^2 + \frac{a}{2\mu\tau}Z^2\right) \tag{3.6}$$

This function is only uniquely defined in the neighbourhood of the fixed point, since it is not 2π-periodic in φ and ψ.

The total derivative of F with respect to τ in the neighbourhood of the fixed point is represented in the form

$$\frac{dF}{d\tau} = -G_0 - \frac{G_1}{\tau} - \frac{G_2}{\tau^2} \tag{3.7}$$

where

$$G_0 = \frac{Y\sin Y(\mu + X)}{aA^2}$$

$$G_1 = \frac{1}{2}\left(X^2 + Y^2 + \left(Y + \frac{Z}{A}\right)^2\right) + o(|\mathbf{R}|^2), \quad G_2 = \frac{aZ^2}{4\mu}; \quad \mathbf{R} = (X, Y, Z)$$

It follows from this that there is a neighbourhood of the fixed point in which $dF/d\tau < 0$ (we recall that, to be specific, we have put $\mu = 0$, $a > 0$) and the fixed point is therefore stable by Lyapunov's theorem. At the same time, as a consequence of the degeneracy of the function G_0, the asymptotic stability requires a more detailed consideration.

We will use the method in Ref. [6]. According to relations (3.6) and (3.7), there is a neighbourhood of the fixed point in which the functions F and G_1 do not have critical points apart from points which satisfy the condition $\mathbf{R} = 0$ and, moreover, $G_1 > 0$ everywhere with the exception of points which satisfy the above-mentioned condition. Consequently, for all initial conditions (when $\tau = \tau_0 > 0$) from this neighbourhood, $F(\tau)$ is a monotonically decreasing function and, since it is bounded from below, there is a limit $F(\tau) \to F^*$ when $\tau \to \infty$. We assume that $F^* > 0$. Comparing equalities (3.6) and (3.7), we conclude that, starting from a certain instant $\tau > \tau_0^* > \tau_0$, the function G_1' will be bounded from below: $G_1 \geq G_1^* > 0$. Integrating relation (3.7), we obtain

$$F^* \leq F(\tau_0) - \int_{\tau_0^*}^{\infty} \frac{G_1^*}{\tau} d\tau$$

However, the last integral diverges and, consequently, the assumption that $F^* > 0$ is untrue, that is, $F^* = 0$. The equilibrium position is therefore asymptotically stable.

Remark 3. The existence of a non-zero function G_0 in relation (3.7) leads to the fact that, along certain directions, the trajectory tends exponentially to the origin of the coordinates.

It has been shown[3] that, in the first approximation with respect to the angle χ, almost all solutions tend to solution (3.2) of the second type. Numerical investigations show that this still holds in the case of arbitrary angles of inclination of the plane.

Hypothesis. Almost all solutions of system (1.2), (3.1) tend to solution (3.2) of the second type which corresponds to the uniformly accelerated sliding of the sleigh along the straight line of steepest descent, during which the centre of mass is located below the of contact point of the blade.

For a numerical analysis of the system, we will prescribe the initial conditions in the $(\varphi, \dot{\varphi} = \omega)$ plane and paint it with different shades of grey depending on the number of revolutions which the body has executed before asymptotically passing to uniform sliding along the line of steepest descent. The greater the number of revolutions, the darker the painted in zone.

The results of the calculations for $v_1 = 0$, $a = 1$, $\mu = 1$, $m = 1.2$, $I = 10$ are shown in Fig. 2, which demonstrates the distribution of the zones corresponding to a different number of revolutions of the sleigh (the enlarged rectangular fragments are shown separately under the numbers 1 and 2). It is seen that the boundaries of these domains can have a complex fractal nature, which corresponds to a complex

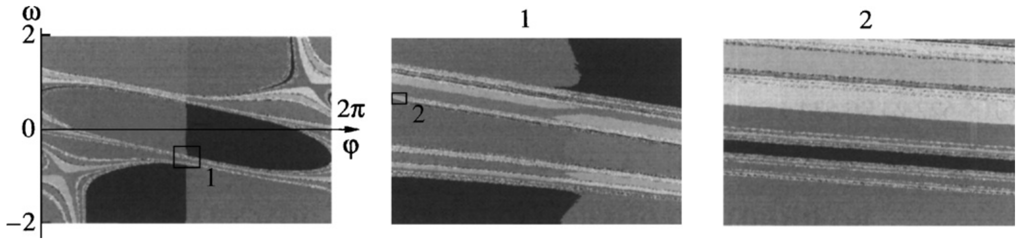

Fig. 2.

structure of the basin of attraction (that is, the sets of initial conditions specifying a motion which asymptotically approaches the fixed point of system (3.4)). This causes indeterminacy in the behaviour of the system, depending on the initial conditions. Such an arrangement of the boundary in this problem appears to be somewhat unexpected since, in the related problem of the fall of a plate in an ideal fluid, the structure of the attraction basins has a regular form[7,8] (that is, the different zones are separated by smooth curves). At the same time the distribution of the initial conditions in the case of the fall of a rigid body of arbitrary shape in an ideal fluid is also of a fractal nature.[9]

Acknowledgements

We wish to thank V. V. Kozlov for useful remarks.

This research was financial by the Russian Foundation for Basic Research (08-01-00651, 07-01-92210). I. S. Mamayev's research was carried out with the help of a grant from the President of the Russian Federation for the support of young scientists/doctors of science (MD-5239.2008.1).

References

1. Chaplygin SA. *Theory of motion of non-holonomic systems. Reducing Factor Theorem. In: collected works*, Vol. 1. Moscow-Leningrad: Gostekhizdat; 1948, 15-25.
2. Bloch A. Asymptotic Hamiltonian Dynamics: the Toda lattice, the three-wave interaction and the nonholonomic Chaplygin sleigh. *Physica D* 2000;**141**(3–4):297–315.
3. Moshchuk NK. The motion of a Chaplygin sleigh. *Prikl Mat Mekh* 1987;**51**(4):546–51.
4. Chaplygin SA. *Collected works. Vol. 3. The Theory of the motion of non-holonomic systems: Examples of the use of the Reducing Factor Method*. Moscow-Leningrad: Gostekhizdat; 1950, 248–59.
5. Neimark YuI, Fufaev NA. *Dynamics of Non-Holonomic Systems*. Moscow: Nauka; 1967.
6. Kozlov VV. *Symmetries, Topology and Resonances in Hamiltonian Mechanics*. Izhevsk: Izd. Udm. Univ; 1995.
7. Kozlov VV. On the fall of a heavy rigid body in an ideal fluid. *Izv Akad Nauk SSSR MTT* 1989;**5**:10–7.
8. Deryabin MV. On asymptotics of the solutions of Chaplygin's equation. *Regulyarnaya khaot/Dinamike* 1998;**3**(1):93–7.
9. Borisov AV, Kozlov VV, Mamayev IS. The fall of a heavy rigid body in an ideal fluid. *Trudy Inst Mat Mekh VrO Ross Akad Nauk* 2006;**12**(1):25–47.

Translated by E. L. S.